# Student seminar exercise sheet Week 2

1. Let $K/F$ be a Galois extension of number fields with Galois group $G$ and $\mathfrak{p} \subset \mathcal{O}_F$ a non-zero prime unramified in $K/F$.

   (a) Let $\mathfrak{P} \subset \mathcal{O}_K$ be a prime dividing $\mathfrak{p}\mathcal{O}_K$. Show that the Frobenius element $\left(\frac{\mathfrak{P}}{K/F}\right)$ at $\mathfrak{P}$ is the unique element $\sigma \in \mathrm{Gal}(K/F)$ with the property $\sigma(\alpha) \equiv \alpha^{N\mathfrak{p}}$ $(mod\ \mathfrak{P})$ for every $\alpha \in \mathcal{O}_K$.

   (b) Assume now that $K/F$ is abelian and let $F \subset L \subset K$ be an intermediate field. Show that

   $$\left(\frac{\mathfrak{p}}{L/F}\right) = \left(\frac{\mathfrak{p}}{K/F}\right)_{|L}.$$

2. Let $K = \mathbb{Q}(\zeta_m)$ be a cyclotomic field and assume without loss of generality that $m$ is odd or divisible by 4.

   (a) Show that $p\mathbb{Z}$ ramifies in $K$ if and only if $p|m$.
   *Hint: Consider the action of the inertia group on $\zeta_m$.*

   (b) Let $p \in \mathbb{Z}$ be a prime not dividing $m$ and $\mathfrak{P} \subset \mathcal{O}_K$ a prime over $p\mathbb{Z}$. Show that if $\zeta, \zeta'$ are $m$-th roots of unity in $K$ such that $\zeta \equiv \zeta'$ $(mod\ \mathfrak{P})$ then $\zeta = \zeta'$.
   *Hint: Differentiate the polynomial $X^m - 1$.*

   (c) Using 1(a), deduce that for any prime $p$ not dividing $m$, the Artin automorphism $\left(\frac{p\mathbb{Z}}{K/F}\right) \in G \cong (\mathbb{Z}/m\mathbb{Z})^\times$ is given by $[p] \in (\mathbb{Z}/m\mathbb{Z})^\times$.

3. For an odd prime $p$, show that the absolute discriminant of $\mathbb{Q}(\zeta_p)$ is given by $(-1)^{\frac{p-1}{2}} p^{p-2}$.

4. Let $d$ be a square-free integer and $K = \mathbb{Q}(\sqrt{d})$ a quadratic extension of $\mathbb{Q}$.

   (a) Show that the absolute discriminant $d_K$ of $K$ is equal to $d$ if $d \equiv 1\ mod\ 4$ and equal to $4d$ if $d \equiv 2, 3\ mod\ 4$.
   *Hint: You may use without proof that*

   $$\mathcal{O}_K = \begin{cases} \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{if } d \equiv 1 \ mod\ 4 \\ \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2,3\ mod\ 4 \end{cases}$$

(b) For $p \in \mathbb{Z}$ an odd unramified prime, show that

$$\left(\frac{p\mathbb{Z}}{K/F}\right) = \left(\frac{d_k}{p}\right),$$

where $\left(\frac{d_k}{p}\right)$ is the Legendre symbol and we identify $\mathrm{Gal}(K/\mathbb{Q})$ with $\{\pm 1\}$.

*Hint: Use Theorem 1.1 from Childress Book*